# THE OPACITY DUE TO COMPTON SCATTERING AT RELATIVISTIC TEMPERATURES IN A SEMIDEGENERATE ELECTRON GAS

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## ABSTRACT

The Rosseland mean due to Compton scattering at temperatures ranging from $kT = 0.01mc^2$ to $kT = 0.35mc^2$ and for densities ranging from $\rho = 10^3$ gm/cm$^3$ to $\rho = 10^6$ gm/cm$^3$ is calculated. In most of this semidegenerate region, Compton scattering dominates over electron conduction, bound-free transitions, and free-free transitions. The results are compared with Sampson's results in the non-degenerate case and his estimates in the semidegenerate case. Upon comparison, his estimates are smaller than ours by approximately 5–15 per cent in most cases.

## INTRODUCTION

Early calculations of the opacity due to Compton scattering have used the Thompson formula for the cross-section, which is valid only for a low temperature and a non-degenerate gas. For stellar-structure studies at high temperatures ($kT \sim mc^2$ or $T \sim 6 \times 10^9$ ° K) and high densities which occur in advanced evolutionary phases, corrections to the Thompson scattering cross-section and to the final state of the electron become important. Sampson (1959) has obtained expressions for the electron-scattering opacity in the non-degenerate relativistic region, taking into account the correction to the Thompson cross-section at high temperatures. In this paper we shall calculate the electron-scattering opacity in the semidegenerate region, in the domain in which the electron opacity is dominant. This domain is shown in Figure 1.

## THEORY OF RADIATION

For completeness, we shall present here a brief derivation of the theory of radiative transfer, taking degeneracy into account. Our derivation is similar to that of Sampson (1959) for the non-degenerate case.

With the assumption of local thermodynamic equilibrium the equation of radiative transfer has the following form in the usual notation:

$$\begin{aligned} \boldsymbol{s}\cdot\nabla I(\nu,\boldsymbol{s}) = & -\mu_a(\nu)[1-\exp(-h\nu/kT)][I(\nu,s)-B(\nu,T)] \\ & -\int_{r_2}\int_{\boldsymbol{P}} N(\boldsymbol{P})\,d\boldsymbol{P}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}(\nu,\boldsymbol{s},\theta,\boldsymbol{P})\,I(\nu,\boldsymbol{s}) \\ & \times\left[1+\frac{c^2}{2h\nu_2{}^3}I(\nu_2,\boldsymbol{s}_2)\right]d\Omega_2+\int_{\Omega_2}\int_{\boldsymbol{P}_2} N(\boldsymbol{P}_2)\,d\boldsymbol{P}_2\left\{1-\left[1+\exp\left(\frac{E-\mu}{kT}\right)\right]^{-1}\right\} \\ & \times\frac{d\sigma}{d\Omega}(\nu_2,\boldsymbol{s}_2,-\theta,\boldsymbol{P}_2)\frac{\nu}{\nu_2}\frac{d\nu_2}{d\nu}I(\nu_2,\boldsymbol{s}_2)\left[1+\frac{c^2}{2h\nu^3}I(\nu,\boldsymbol{s})\right]d\Omega_2\,, \end{aligned} \tag{1}$$

where $I(\nu,s)$ is the intensity of radiation of frequency $\nu$ traveling in the direction of the unit vector $\boldsymbol{s}$,$B(\nu,T)$ the equilibrium radiation intensity given by

$$B(\nu,T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}-1\right)\right]^{-1}; \tag{2}$$

and $N(\boldsymbol{P})$, the electron distribution function given by

$$N(\boldsymbol{P})\,d\boldsymbol{P}=\frac{2}{h^3}\frac{P^2}{1+\exp[(E-\mu)/kT]}\,dP d\Omega_P\,. \tag{3}$$

In equation (1) the subscript "2" refers to the final states of the electron and the photon. By principle of detailed balance, we obtain

$$\begin{aligned} N(\boldsymbol{P}_2)\,d\boldsymbol{P}_2\left\{1-\left[1+\exp\left(\frac{E-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega}\frac{\nu}{\nu_2}\frac{d\nu_2}{d\nu} \\ =N(\boldsymbol{P})\,d\boldsymbol{P}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}\left(\frac{\nu}{\nu_2}\right)^3\exp\frac{h\nu_2-h\nu}{kT}\,. \end{aligned} \tag{4}$$

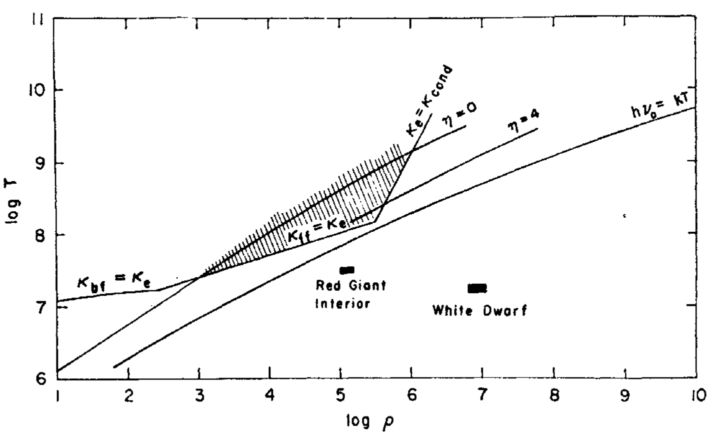


FIG. 1.—Temperature-density diagram. The shaded area shows the region under consideration and $\nu_0$ stands for plasma frequency.

Substituting equation (4) into equation (1) we obtain

$$\begin{aligned} \boldsymbol{s}\cdot\nabla I(\nu,\boldsymbol{s}) &= -\mu_a(\nu)[1-\exp(-h\nu/kT)][I(\nu,\boldsymbol{s})-B(\nu,T)] \\ &-\int_P\int_{\Omega_2}N(\boldsymbol{P})\,d\boldsymbol{P}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}(\nu,\boldsymbol{s},\theta,\boldsymbol{P})\left\{I(\nu,\boldsymbol{s})\right. \\ &\left.\times\left[1+\frac{c^2}{2h\nu_2{}^3}I(\nu_2,\boldsymbol{s})\right]-I(\nu_2,\boldsymbol{s}_2)\left[1+\frac{c^2}{2h\nu^3}I(\nu,\boldsymbol{s})\right]\left(\frac{\nu}{\nu_2}\right)^3\exp\frac{h\nu_2-h\nu}{kT}\right\}d\Omega_2\,. \end{aligned} \tag{5}$$

Assuming the solution to equation (5) to be

$$I(\nu,s)=B(\nu,T)-l(\nu)\boldsymbol{s}\cdot\nabla B(\nu,T)+\ldots, \tag{6}$$

we obtain

$$l(\nu)=\{\mu_a(\nu)[1-\exp(-h\nu/kT)]+\mu_s(\nu)\}^{-1}, \tag{7}$$

where

$$\begin{aligned} \mu_s(\nu)=\int_P\int_{\Omega_2}N(\boldsymbol{P})\,d\boldsymbol{P}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\} \\ \times\frac{d\sigma}{d\Omega_2}\frac{1-\exp(-h\nu/kT)}{1-\exp(-h\nu_2/kT)}\left[1-\frac{\nu_2 l(\nu_2)}{\nu l(\nu)}\cos\theta\right]d\Omega_2\,, \end{aligned} \tag{8}$$

where $\theta$ = scattering angle of the photon.

In equation (8) we have already dropped those terms which vanish upon integration over $\Omega$ to obtain the radiation flux. We now use the following approximations

$$\frac{1-\exp(-h\nu/kT)}{1-\exp(-h\nu_2/kT)} \simeq 1 \tag{9}$$

and

$$\frac{l(\nu_2)}{l(\nu)} \simeq 1 . \tag{10}$$

Since the main contribution to the integrals comes from photons with $h\nu \simeq 4kT$, and $\langle \nu_2 \rangle$ is not very different from $\nu$, the approximation (9) is very good. For Compton scattering $l(\nu)$ changes very slowly with the frequency $\nu$. For instance, at $kT = 0.05mc^2$ and $\eta = +1$, $\mu_s(h\nu = 4.125kT)/\mu_s(h\nu = 3.208kT) = 0.957$. Thus the error introduced in replacing $\{1 - \nu_2 l(\nu_2) \cos\theta/[\nu l(\nu)]\}$ by $(1 - \nu_2 \cos\theta\, \nu^{-1})$ is of the order of a few per cent.

With the above approximations, equation (8) is reduced to

$$\mu_s(\nu) = \int_P \int_{\Omega_2} N(\boldsymbol{P})\, d\boldsymbol{P} \left\{1 - \frac{1}{1+\exp[(E_2-\mu)/kT]}\right\} \frac{d\sigma}{d\Omega_2}\left(1 - \frac{\nu_2}{\nu}\cos\theta\right) d\Omega_2 . \tag{11}$$

Frank-Kamensetskii (1962) called $(d\sigma/d\Omega_2)(1 - \nu_2 \cos\theta/\nu)$ the transport cross-section.

In the classical limit, $\nu_2$ is equal to $\nu$ and the differential cross-section is an even function of $\cos\theta$. Hence, upon integration over $\Omega_2$, the contribution from $\nu_2 \cos\theta/\nu$ will be zero and the transport cross-section is equal to the scattering cross-section.

The electron distribution function is given by

$$N(\boldsymbol{P})\, d\boldsymbol{P} = \frac{2}{h^3} \frac{P^2}{1+\exp[(E-\mu)/kT]}\, dP d\Omega_P . \tag{12}$$

In the degenerate region the contribution to the density function by the creation of positrons is small. The correction due to positron creation will be shown to be small.

## CALCULATIONS

To facilitate computation we introduce the following non-dimensional variables

$$\begin{aligned} E' &\equiv E/mc^2 - 1 , & T' &\equiv kT/mc^2 , \\ u &\equiv h\nu/kT , & u_2 &\equiv h\nu_2/kT . \end{aligned} \tag{13}$$

In terms of these variables equations (12) and (11) can be expressed as

$$N(\boldsymbol{P})\, d\boldsymbol{P} = \frac{2}{\lambda_c^3} \frac{(E'^2 + 2E')^{1/2}(1+E')}{1+\exp(E'/T' - \eta)}\, dE' d\Omega_P \tag{14}$$

and

$$\begin{aligned} \mu_s(u, kT, \eta) = \frac{2}{\lambda_c^3} \int_{E'=0}^{\infty} \int_{\Omega_P} \int_{\Omega_2} \frac{(E'^2 + 2E')^{1/2}(1+E')}{1+\exp(E'/T' - \eta)} \\ \times \left[1 - \frac{1}{1+\exp(E'/T' + u - u_2 - \eta)}\right] \frac{d\sigma}{d\Omega_2}\left(1 - \frac{\nu_2}{\nu}\cos\theta\right) d\Omega_2 d\Omega_P dE' , \end{aligned} \tag{15}$$

where $\lambda_c$ is the Compton wavelength $= h/mc$. In obtaining equation (15) we have eliminated $E_2$ by means of the energy conservation equation $E + h\nu = E_2 + h\nu_2$.

The direction of the incident electron is chosen as the positive $z$-axis. The direction of the incident photon is chosen to lie in the $x$–$z$-plane and it makes an angle $\alpha$ with the

$z$-axis. The direction of the final photon has the angular coordinates $(\alpha', \varphi)$ as shown in Figure 2. In the coordinate system so chosen, we have

$$d\Omega_2 = d\varphi d \cos \alpha' , \qquad d\Omega_P = d\Omega_\nu = 2\pi d \cos \alpha . \tag{16}$$

Then equation (15) becomes

$$\mu_s(u, kT, \eta) = \frac{8\pi}{\lambda_c^3} \int_{E'=0}^{\infty} dE' \int_{\cos \alpha=-1}^{1} d \cos \alpha \int_{\cos \alpha'=-1}^{1} d \cos \alpha' \int_{\varphi=0}^{\pi} d\varphi$$
$$\times \frac{(E'^2+2E')^{1/2}(1+E')}{1+\exp(E'/T'-\eta)} \left\{ 1 - \left[ 1 + \exp\left( \frac{E'}{T'} + u - u_2 - \eta \right) \right]^{-1} \right\} \frac{d\sigma}{d\Omega_2} \left( 1 - \frac{\nu_2 \cos \theta}{\nu} \right). \tag{17}$$

The scattering angle $\theta$ is easily shown to be given by

$$\cos \theta = \cos \alpha \cos \alpha' + \sin \alpha \sin \alpha' \cos \varphi . \tag{18}$$

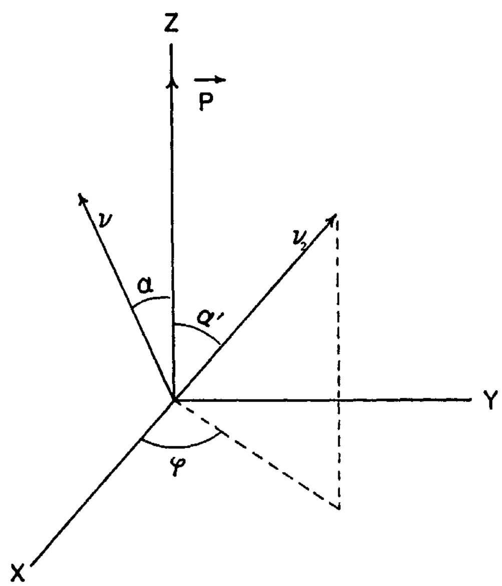


FIG. 2.—Coordinate system used in the calculation

By the conservation laws $\nu_2$ is related to $\nu$ and the other variables through

$$\frac{\nu_2}{\nu} = \frac{1 - \beta \cos \alpha}{1 - \beta \cos \alpha' + h\nu(1 - \cos \theta)/E'}, \tag{19}$$

i.e.,

$$\frac{u_2}{u} = \frac{1 - \beta \cos \alpha}{1 - \beta \cos \alpha' + uT'(1 - \cos \theta)/(1 + E')} . \tag{20}$$

We use a form of the differential cross-section given by Jauch and Rohlich (1955):

$$\frac{d\sigma}{d\Omega_2} = \frac{r_0^2}{2r^2} \left( \frac{\nu_2}{\nu} \right)^2 \frac{X}{(1 - \beta \cos \alpha)^2}, \tag{21}$$

where $r_0$ = classical radius of the electron and

$$X = \frac{\nu}{\nu_2} \frac{1-\beta \cos \alpha}{1-\beta \cos \alpha'} + \frac{1-\beta \cos \alpha'}{1-\beta \cos \alpha} \frac{\nu_2}{\nu} - 1$$

$$+ \left[ 1 + \frac{m^2 c^4}{Eh\nu(1-\beta \cos \alpha)} - \frac{m^2 c^4}{Eh\nu_2(1-\beta \cos \alpha')} \right]^2 . \tag{22}$$

Thus, substituting equations (18)–(22) into equation (17), we can compute $\mu_s(u, kT, \eta)$. The Rosseland mean is given by

$$\Lambda(kT, \eta) = \frac{15}{4\pi^4} \int_0^\infty \frac{1}{\mu_s(u, kT, \eta)} \frac{u^4 e^u}{(1-e^u)^2} du . \tag{23}$$

We have neglected $\mu_a$ in the scattering-dominated region. In all, we have a fivefold integration to do for each of the corresponding values of $kT$ and $\eta$. The computation was done on an IBM 9094 computer.

TABLE 1

VALUES OF $G(kT,\eta)$ FOR VARIOUS TEMPERATURES AND DEGENERACY PARAMETERS $\eta$

| $kT/(mc^2)$ | $\eta$ | | | | | |
|---|---|---|---|---|---|---|
| | −"∞" | −1 | 0 | 1 | 2 | 4 |
| 0 01 | . | 0 8294 | 0 7368 | 0 6112 | 0 4864 | 0 3094 |
| 03 | . . . | .6732 | .6060 | .5094 | 4076 | 2554 |
| 05 | 0 5967 | .5699 | .5189 | .4410 | 3553 | 2217 |
| 15 | 3496 | .3416 | .3194 | .2816 | 2322 | 1423 |
| 25 | 0 2542 | .2529 | .2389 | .2135 | 1776 | 1066 |
| 0 35 | . | 0.2028 | 0.1923 | 0 1730 | 0 1443 | 0 0843 |

Defining $G(kT,\eta)$ as the transport cross-section in units of the Thompson cross-section $\sigma_0$, then the Rosseland mean is

$$\Lambda(kT, \eta) = \frac{1}{\sigma_0 N G(kT, \eta)} , \tag{24}$$

where $N$ is the number of electrons given by equation (14). The opacity is $\kappa = (\rho\Lambda)^{-1}$, where $(\sigma_0 N)^{-1}$ is the Rosseland mean in the classical limit. Values of $G(kT,\eta)$ are tabulated in Table 1 for the corresponding values of $kT$ and $\eta$. The values corresponding to $\eta$ = −"∞" were obtained from

$$\bar{G}(T) = -0.13887 + 4.9871(kT)^{-1/2} - 5.9479(kT)^{-1}$$

$$- 2.362(kT)^{-3/2} (\text{for } 20 \text{ keV} \leq kT \leq 125 \text{ keV}) \tag{25}$$

from Sampson (1959) for the non-degenerate case where $kT$ is measured in keV.

If the values of $G(kT,\eta)$ at other $kT$ and $\eta$ are needed, the following polynomial of

$kT$ and $\eta$ will fit fairly well for the region under consideration with an error of approximately 5 per cent,

$$\log_e G(kT,\eta) = -0.3037 - 6.89757\frac{kT}{mc^2} + 8.89771\left(\frac{kT}{mc^2}\right)^2 - 0.158737\eta$$
$$+0.392553\frac{kT}{mc^2}\eta - 0.0146867\eta^2 - 0.451961\left(\frac{kT}{mc^2}\right)^2\eta - 0.0523759\frac{kT}{mc^2}\eta^2. \qquad (26)$$

To facilitate application of the transport cross-section due to Compton scattering, we list the number densities of the electron as given by equation (14) in Table 2.

TABLE 2

VALUES OF $N_e\lambda_c^3$ AT VARIOUS VALUES OF $kT$ AND $\eta$

| $kT/(mc^2)$ | $\eta$ | | | | |
|---|---|---|---|---|---|
| | −1 | 0 | +1 | +2 | +4 |
| 0 01 | 0 01046 | 0 02448 | 0 05061 | 0 09077 | 0 2117 |
| 03 | 0 05650 | 0 1326 | 0 2755 | 0 4974 | 1 183 |
| .05 | 0 1264 | 0 2972 | 0 6195 | 1 129 | 2 727 |
| .15 | 0 7847 | 1 865 | 3 963 | 7 442 | 19 25 |
| 25 | 1 983 | 4 754 | 10 25 | 19 69 | 53 45 |
| 0 35 | 3 804 | 9 180 | 20 03 | 39 14 | 110 2 |

## DISCUSSION

We have so far neglected the positron contribution to Compton scattering. It can be shown that at $\eta = -1$ and $kT = 0.35mc^2$, which are most favorable to pair creation in the region under consideration, the ratio of the number density of electrons and positrons to that of electrons is 1.027. At all other values of $\eta$ and $kT$ in Table 2 the ratio is less than 1.005. While the correction to the number density arising from the presence of positrons at $\eta = -1$ and $kT = 0.35mc^2$ is +2.7 per cent, the corresponding correction to effective cross-section $G(kT,\eta)$ should be much less than that since the latter is less sensitive to the presence of positrons than is the number density $N$.

If our computation is repeated for $\eta = -3$, the results agree with Sampson's (1959) to within 2 per cent.

Sampson (1961) suggested that one estimate the effective degenerate cross-section by multiplying his non-degenerate results by a factor

$$Q(T',\eta) = \left\{\int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}dE}{[1+\exp(E/T'-\eta)][1+\exp(\eta-E/T')]}\right\}$$
$$\times\left\{\int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}dE}{[1+\exp(E/T'-\eta)]}\right\}^{-1}. \qquad (27)$$

Upon comparison his estimates are smaller by approximately 5–15 per cent than ours in most cases.

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